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On improving pension product design

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Abstract

The paper provides some guidelines to individuals with defined contribution (DC) pension plans on how to manage pension savings both before and after retirement. We argue that decisions regarding investment, annuity payments, and the size of death sum should not only depend on the individual's age (or time left to retirement), nor should they solely depend on the risk preferences, but should also capture: 1) economical characteristics - such as current value on the pension savings account, expected pension contributions (mandatory and voluntary), and expected income after retirement (e.g. retirement state pension), and 2) personal characteristics - such as risk aversion, lifetime expectancy, preferable payout profile, bequest motive, and preferences on portfolio composition. Specifically, the decisions are optimal under the expected CRRA utility function and are subject to the constraints characterizing the individual.

The problem is solved via a model that combines two optimization approaches: stochastic optimal control and multi-stage stochastic programming. The first method is common in financial and actuarial literature, but produces theoretical results. However, the latter, which is characteristic for operations research, has highly practical applications. We present the operations research methods which have potential to stimulate new thinking and add to actuarial practice.

1 Introduction

Recent years have seen a decided worldwide shift from defined benefits (DB) pension plans towards defined contribution (DC). The number of participants in DC plans is quickly expanding because these plans are not only easier and cheaper to administer, but also more transparent and more flexible. Furthermore they can better capture the individual's needs. However, a primary problem is that the participants often do not know how to manage their saving and investment decisions.

In some countries, such as the U.S., most DC decisions are made by the individual with little advice from the employer. In contrast, in countries such as Denmark, the sponsoring organizations, including life insurers, suggest a dynamic investment strategy suitable to the individual's age and risk preferences. Individuals in most of the countries also have to decide on how to spend the amount accumulated on their pension savings account. Should they follow a certain withdrawal rate rule, or should they purchase annuities that will provide with regular payments during retirement? This task is not easy, especially when life insurers offer a wide variety of annuity products (e.g. fixed or variable, deferred or immediate, term or whole-life). How can the individuals know, which product is best for them?

There is one more decision they have to keep in mind. Namely, what to do with the savings in case of their death? Do they want to bequeath the savings to their heirs, or maybe purchase an annuity product combined with a life insurance policy? What level of death sum should they choose?

We argue that aforementioned decisions should differ for each individual and should account for the following factors: 1) economical characteristics - such as current value on the pension savings account, expected pension contributions (mandatory and voluntary), and expected income after retirement (e.g. retirement state pension), and 2) personal characteristics - such as risk aversion, lifetime expectancy, preferable payout profile, bequest motive, and preferences on portfolio composition.

To help the individuals manage the savings and investment decisions we build an optimization-based financial planning model. Because such a model can be complicated and difficult to solve, we propose to combine two popular methodologies: multi-period stochastic programming (MSP) and stochastic optimal control (SOC), also referred to continuous-time and state dependent dynamic programming. The latter method is common in financial and actuarial literature, and, although best applied for simple models, provides the intuition behind the optimal solutions. See for example, (Yaari, 1965), (Samuelson, 1969), (Merton, 1969, 1971) and (Richard, 1975), for optimal decisions regarding investment, consumption and sum insured.

On the contrary, MSP, which is characteristic for operations research, has highly practical application and complement SOC approach, especially in terms of adding realistic constraints and modeling more complicated processes. In stochastic programming approach we model the possible outcomes for the uncertainties in a scenario tree, and numerically compute the optimal solution at

each node of the tree. See, for example, (Carino et al., 1998) and (Carino and Ziemba, 1998), who formulate a financial planning model for one of the biggest Japanese property and casualty insurer, (Mulvey et al., 2003), who present a multi-period stochastic network model for integrating corporate financial and pension planning, and (Mulvey et al., 2008) who expand this work by adding the borrowing decisions. The applications of MSP to individual asset-liability management can be found, for example in (Ziemba and Mulvey, 1998), (Kim et al., 2012) and (Konicz and Mulvey, 2013). However, the main drawback of this optimization method is the ability to handle many periods under enough uncertainty. Especially, modelling the entire lifetime of an individual is challenging in terms of computational tractability.

To benefit from both optimization approaches and to avoid the aforementioned drawbacks, we combine them into one mathematical framework. We solve the problem using MSP approach up to some horizon T , and to ensure that the model accounts for the entire lifetime of an individual, we insert the *end effect* in the objective function of MSP. The end effect is determined by the optimal value function calculated explicitly via SOC technique. This function covers the period from the horizon T to the individual's death. Combining these two optimization approaches is new and has only been investigated in (Geyer et al., 2009) and (Konicz et al., 2013).

The paper is organized as follows. Section 2 describes the economical and personal characteristics that we take into account when advising on how to manage the pension savings. Section 3 presents the financial planning model. Section 4 explains the intuition behind the optimal solution obtained from MSP model. Section 5 includes numerical examples illustrating the application of the model for different individuals. Section 6 concludes. Finally, Appendix A introduces multi-stage stochastic programs and Appendix B presents details of the explicit solution derived via SOC approach. We argue that management of savings in DC pension plan should account for economical and personal characteristics, and it should be tailored to a customer. Our model takes into account the following factors.

2 Economical and personal characteristics

We argue that management of savings in DC pension plan should account for economical and personal characteristics, and it should be tailored to a customer. Our model takes into account the following factors.

2.1 Economical characteristics

Current value on the savings account The value of the individual's account, X_t , develops according to the initial savings x_0 , contributed premiums, capital gains including dividends, insurance coverage, accredited survival credit and the benefits paid after retirement - all these elements are described below.

Premiums Until retirement the individual contributes to the savings account. The premiums P_t^{tot} consist of a fixed percentage, p^{fixed} , of the labor income, l_t , that, in many countries, is mandatory and decided by the employer, and the additional voluntary contributions, $p^{vol} l_t$. The latter may be of interest of an individual who wishes to increase the future benefits.

$$P_t^{tot} = (p^{fixed} + p^{vol})l_t, \quad p^{fixed} \in [0, 1], \quad p^{vol} \in [0, 1 - p^{fixed}].$$

The labor income l_t is deterministic and increases with a salary growth rate y_l , $l_t = l_0 e^{y_l t}$, where l_0 is the level of the labor income at the current time t_0 . Both the premiums and the labor income are positive only until retirement, $t < T_R$; otherwise 0.

State retirement pension After retirement the individual has no other income than state retirement pension, b_t^{state} . This income is typically financed on a pay-as-you-go basis from general tax revenues, and ensures a basic standard of living for old age. It often depends on the level of the individual's income before retirement, but not on the income from the DC plan. We assume that the state retirement pension consists of the life long, yearly adjusted payments.

2.2 Personal preferences

Risk aversion The individual is risk averse and has a utility function u characterized by a constant relative risk aversion (CRRA), $1 - \gamma$, and the time dependent weights w_t . The impatience factor ρ , which is included in function w_t , reflects the importance of the benefits and death sum now relatively to how important these payments would be in the future:

$$u(t, B_t) = \frac{1}{\gamma} w_t^{1-\gamma} B_t^\gamma, \quad w_t = e^{-\frac{\rho t}{1-\gamma}}, \quad \forall \gamma \in (-\infty, 1) \setminus \{0\}.$$

The choice of $\gamma = 0$ implies the logarithmic utility.

Lifetime expectancy The individual has uncertain lifetime, which we model with two kinds of mortality rates: μ_t and ν_t . The first function is the subjective mortality rate and reflects the individual's opinion about her health status and lifestyle. For example, does she live a healthy lifestyle and thus expect to live longer than others? Is she a regular smoker or maybe seriously ill? The answers to these questions affect the decisions regarding the payout profile as well as the decision about purchasing life insurance.

The second function, ν_t , also referred to pricing mortality, is used by life insurers for calculating the price of their life contingent products. Especially in European countries, due to legislation, both the survival credit and the price for life insurance are calculated under unisex criteria, and the individual is not even subject to health screening, see (Rocha et al., 2010). A person with a cancer disease, heart attack, a regular smoker or an overweight person has the right to the same benefits

as a healthy individual. Pricing mortality rates are typically reported in the actuarial life tables.

Payout profile Upon retirement the individual starts receiving the retirement income. The benefits come from two sources: the state retirement pension, b_t^{state} , and the benefits from the DC plan, B_t :

$$B_t^{tot} = b_t^{state} + B_t, \quad t \geq T_R.$$

The benefits from the DC plan, also known as the labor market pension (occupational pension, 2nd pillar) and the individual retirement accounts (private pension, 3rd pillar), are one of the most important decision variables in the optimization problem. Individuals can choose among the following possibilities:

- *Duration of the payments.* Is the individual interested in receiving a lump sum benefit upon retirement, regular payments over a period of 10 or 25 years (term annuity), or regular payments as long as she is alive (life annuity)? We can control the duration in the model by choosing the appropriate value of \tilde{T} , which denotes the time of receiving the last benefit.
- *Payout curve.* If interested in annuities, does she prefer to receive constant, increasing or maybe decreasing payments? By inserting the right parameters in the utility function: the impatience factor ρ and risk aversion $1-\gamma$, we can control the payout curve. Another important factor to consider is the lifetime expectancy, which depends on the subjective mortality rate μ_t . A person with health problems might want to spend more of her savings during the first years of retirement, whereas a person who expects to live long would want to make sure she would never outlive her resources.
- *Size of the payments.* To increase the size of the benefits, the individual can either increase the premiums or choose a more aggressive investment strategy, for example by choosing equity-linked (variable) annuities, as defined e.g. in (Blake et al., 2003). These products differ from traditional fixed rate annuities that offer a constant level of payments during retirement in a way, that their size is regularly adjusted to account for capital gains and losses. This adjustment is necessary to avoid the danger of running out of the resources before death. However, equity-linked annuities are directly linked to market returns, therefore can be risky.

Bequest motive Not surprisingly, the marital status and dependants play a crucial role in the choice of the pension product. A single individual would be interested in a life annuity. In this type of contract an individual agrees to give up the savings upon death, which are then inherited by the life insurer. Thus, in contrast to other products, life annuities provide with an additional return arising from mortality risk sharing. This return is often called a survival credit and is proportional to the value of the individual's savings, i.e. $\nu_t X_t$.

The individual with dependants would bequeath some death sum Beq_t to her heirs. We assume that upon death, life insurer inherits the value of the savings, but pays out Beq_t to the dependants. The size of death benefit can be proportional to the value of the account through a factor ins_t ,

$$Beq_t = ins_t X_t,$$

or can be a decision variable in the program. In the latter case, we introduce an additional parameter k , which defines the strength of the bequest motive relatively to the received benefits. The actuarially fair price for the insurance coverage is equal to $\nu_t Beq_t$. Because choosing larger death benefit leads to lower annuity payments, and vice versa, it is not easy to find the right balance between the level of annuity payments and the level of death sum.

Portfolio composition Retirement savings can be allocated to a number of financial assets. In the U.S. the individuals have lots of flexibility and can invest in highly leveraged financial products. In countries such as Denmark, the individuals have a limited list of assets to choose from. For example, life insurers offering unit-linked products allow for investments within their own list of mutual funds and ETFs, which replicate stock indices for different regions and industries, corporate and government bonds with different maturities, commodities, etc.

Our model allows the individual to include her preferences regarding asset allocation. For simplicity, we consider portfolios composed of positions in four asset classes: cash (corresponding to a 3-month short rate), an aggregate bond index including both government and corporate bonds with different durations, a domestic stocks index and an international stocks index.

3 Multi-stage stochastic program formulation

Stochastic programming is a general purpose framework for modeling optimization problems. We include a brief introduction to stochastic programming in Appendix A, whereas more details can be found in the classical books on the subject, for example (Birge and Louveaux, 1997), (Zenios, 2008), and (Shapiro et al., 2009).

The range of possible outcomes for the uncertainties is modeled by a *scenario tree*, which consists of nodes $n \in \mathcal{N}_t$ uniquely assigned to stages $t = 1, \dots, T$. Each node has a probability $prob_n$, so that $\forall_t \sum_{n \in \mathcal{N}_t} prob_n = 1$. At the first stage we have only one root node n_0 , whereas the number of nodes at other stages corresponds to possible values of random vector ξ_t . Every node $n \in \mathcal{N}_t, t > t_0$, has a unique ancestor n^- , and every node $n \in \mathcal{N}_t, t < T$, has children nodes n^+ . The nodes with no children are called the leaves. A scenario \mathcal{S}^n is a set of all predecessors of a leaf $n : n^-, n^{--}, \dots, n_0$, or equivalently, a single branch from the root to the leaf. The number of scenarios in the tree equals the number of leaves. Fig. 1 illustrates an example of a three-stage scenario tree.

A considerable amount of literature focuses on scenario generation methods for stochastic pro-

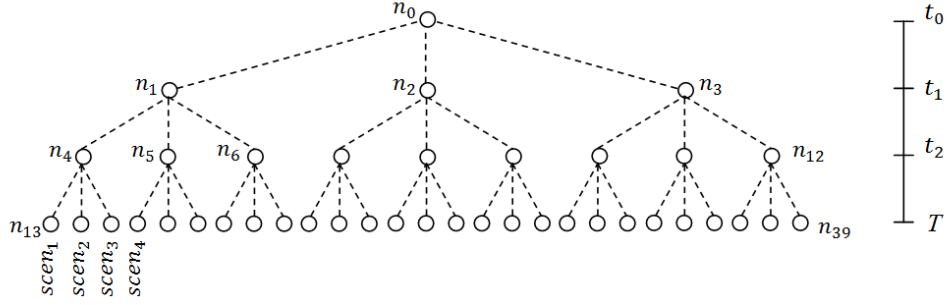


Figure 1: A 3-period scenario tree with a constant branching factor 3 and $3^3 = 27$ scenarios.

gramming. Among different approaches we can distinguish sampling, simulation, scenario reduction techniques and moment matching methods. For the purpose of our study we have chosen the technique that matches the statistical properties (the first four moments and the correlations) of the underlying processes. This approach has been introduced by (Høyland and Wallace, 2001), who suggest solving a nonlinear optimization problem that minimizes the distance between the properties of the generated tree and of the underlying process. Both the asset returns and the probabilities of each node are the decision variables in this formulation.

During recent years several authors have been investigating possible improvements of the moment matching approach. (Ji et al., 2005) show that if one can predetermine the outcomes of the asset returns (e.g. by simulation) and choose the probabilities of the nodes to be the only variables in the model, then it is possible to match the statistical properties of the underlying process with a linear optimization problem. This method is further improved by (Xu et al., 2012) who combine the simulation, the K -means clustering approach, and the linear moment matching, and by (Chen and Xu, 2013), who remove the simulation component and applies the K -means clustering method directly onto the historical dataset.

Once generating the scenario tree with events of death and asset returns, (the parameters for calculation of the survival probabilities and for the asset returns distribution are given in Table 4), we can calculate the savings and investment decisions. The optimal solution depends on the possible future realizations of the asset returns and death events, and on the decisions made in the previous stage. Fig. 2 shows a fragment of a multi-stage tree with numerically calculated asset allocation, annuity payments and death sum. To obtain the entire tree, for each period $t \in [t_0, T]$, node $n \in \mathcal{N}_t$, and asset class $i \in \mathcal{I}$, we define the following parameters and decision variables:

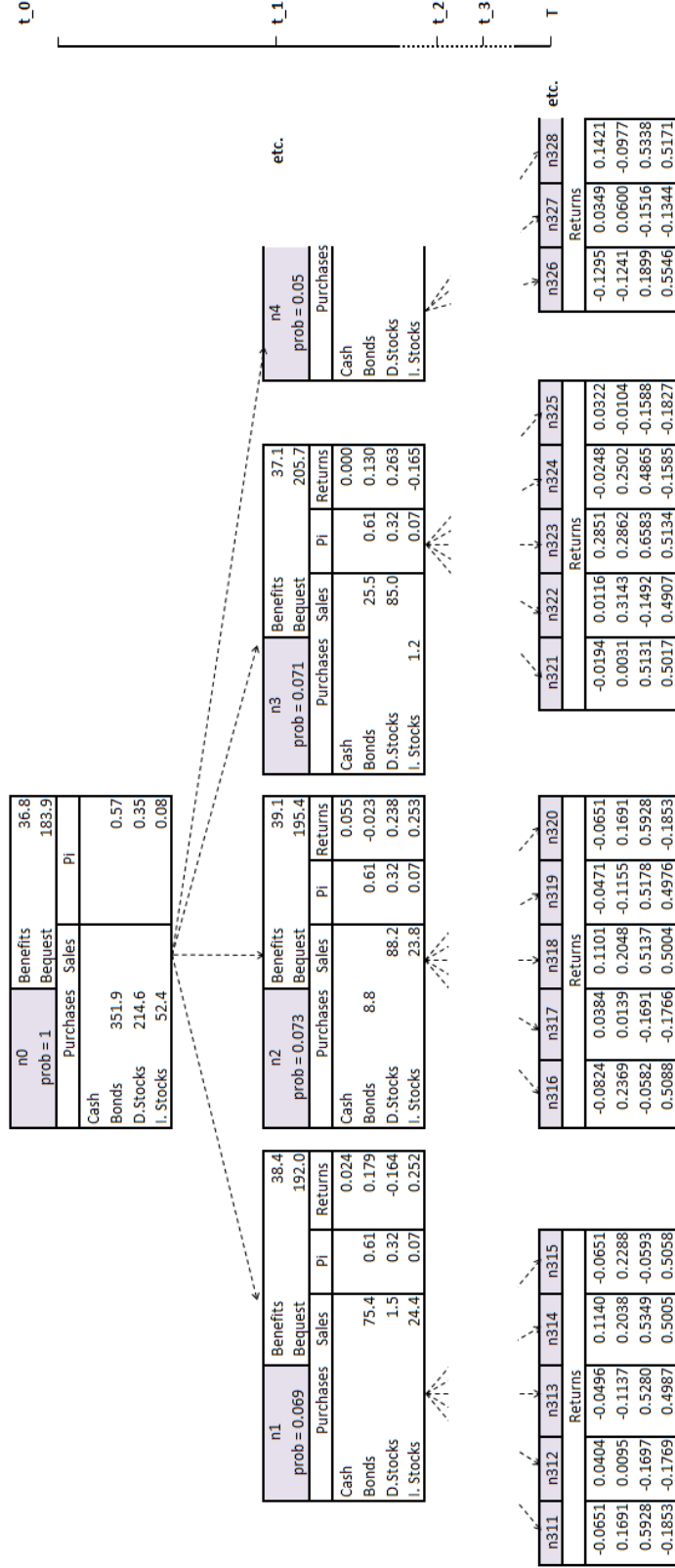


Figure 2: A fragment of the multi-stage problem. The total benefits, the size of death benefit and the rebalancing decisions (purchases and sales) are calculated numerically at each node. P_i denotes the proportion in each asset class. The periods between the decisions are of $\Delta t = \{1, 3, 3\}$, thus the returns at t_0 are the yearly returns, whereas the returns in the following periods are accumulated over 3 years. The amounts are in EUR 1,000. Other parameters: $age_0 = 65$, $x_0 = 650$, $b_t^{state} = 4$, $\gamma = -4$, $\rho = 0.04$, $k = 5$, $d_i = 0$, $u_i = 1$, $T = 10$, and $\Delta t = \{1, 3, 3, 3\}$.

Parameters

T_R	retirement time
T	the end of decision horizon and the beginning of the remaining period modeled by the end effect
\tilde{T}	expiration of the contract, i.e. the last benefit is paid out
$prob_n$	probability of being at node n
x_0	initial value of savings
x_T^{min}	minimum level of savings upon horizon T
b_t^{state}	state retirement pension at time t
b_t^{min}	minimum level of benefits at time t
l_t	labor income at time t
p^{fixed}	fixed percentage of the labor income defining the mandatory premiums
p^{vol}	fixed percentage of the labor income defining the maximum voluntary premiums
ins_t	proportion of the savings defining the death sum
k	weight on the bequest motive relatively to the size of the benefits
$t\tilde{p}_y$	probability that an y -year old individual survives to at least age $y + t$ (individual's expectation)
tp_y	probability that an y -year old individual survives to at least age $y + t$ (insurer's expectation)
\tilde{q}_y	probability that an y -year old individual dies during the following period (individual's expectation)
q_y	probability that an y -year old individual dies during the following period (insurer's expectation)
$r_{i,t,n}$	return on asset i at node n corresponding to stage t .

Decision variables

$X_{i,t,n}^{\rightarrow}$	amount allocated to asset class i , at the beginning of period t , at node n , before rebalancing and any cash-flows
$X_{i,t,n}$	amount allocated to asset class i , at the beginning of period t , at node n , after rebalancing and any cash-flows
$X_{i,t,n}^{buy}$	amount of asset class i purchased for rebalancing in period t , at node n
$X_{i,t,n}^{sell}$	amount of asset class i sold for rebalancing in period t , at node n
$B_{t,n}$	benefits (annuity payments) from the DC pension plan received in period t , node n
$B_{t,n}^{tot}$	total benefits paid in period t , node n
$P_{t,n}^{tot}$	total premiums (mandatory and voluntary) paid in period t , node n
$Beq_{t,n}$	death sum paid to the heirs upon the individual's death in period t , node n .

All the savings are initially allocated to cash (denoted by asset class $i = 1$), thus $X_{1,t_0,n_0}^{\rightarrow} = x_0$ and $X_{i,t_0,n_0}^{\rightarrow} = 0, \forall i \neq 1$. The administration costs, transaction costs, and the taxes have been ignored for simplicity.

The objective function, eq. (1), which we aim to maximize, consists of three terms: (i) the expected utility of total retirement benefits paid while the person is alive, (ii) the expected utility of death sum paid to the heirs upon the individual's death, and (iii) the end effect described in detail in Appendix B. The budget constraint, eq. (2), specifies the cash-flows accompanying the savings account: the incoming payments (capital gains, amount gained from the sales of the securities,

premiums, and survival credit) and the outgoing payments (the amount spent on the purchase of new securities, annuity payments, and insurance coverage). The next constraint, (3), defines the asset inventory balance. We first account for the returns earned during the previous period, eq. (4), and then rebalance the amount by purchasing or selling a given asset. In (5) we define the total premiums paid to the savings account as the sum of the mandatory and voluntary contributions. Constraint (6) defines the total benefits as the sum of the state retirement pension and the annuity payments received from the DC plan. By including equations (7) and (8), we ensure that the benefits and the value of the savings do not fall below the certain pre-specified levels b_t^{min} and x_T^{min} , respectively. Constraint (9) defines the death benefit as a fraction of the savings, whereas (10) defines the actuarially fair survival credit that the individual receives for each period she survives. If the individual wishes to bequeath exactly the value of the savings, i.e. $ins_t = 1$, then the survival credit is equal to the price of the death benefit, and the last two terms in the budget constraint (2) cancel out. In a case when the individual is interested in the optimal death sum, constraint (9) is no longer necessary and should be removed. Equations (11)-(12) define the limits on portfolio composition. These can reflect the regulatory constraints, for example, no shorting and gearing is allowed ($u_i = 1$ and $d_i = 0$), or they can reflect the individual's preferences on portfolio composition. Finally, we include eq. (13) to distinguish between the purchases and sales, and to ensure that the annuity payments and the death sum are positive.

$$\begin{aligned} \text{maximize} \quad & \sum_{s=\max(t_0, T_R)}^{T-1} \sum_{n \in \mathcal{N}_s} s \tilde{p}_y u(s, B_{s,n}^{tot}) \cdot prob_n + \sum_{s=t_0}^{T-1} \sum_{n \in \mathcal{N}_s} s \tilde{p}_y \tilde{q}_{y+s} k u(s, Beq_{s,n}) \cdot prob_n \\ & + T \tilde{p}_y \sum_{n \in \mathcal{N}_T} V \left(T, \sum_i X_{i,T,n}^{\rightarrow} \right) \cdot prob_n, \end{aligned} \quad (1)$$

subject to

$$X_{1,t,n} = X_{1,t,n}^{\rightarrow} + \sum_{i \neq 1} X_{i,t,n}^{sell} - \sum_{i \neq 1} X_{i,t,n}^{buy} + P_{t,n}^{tot} \mathbf{1}_{\{t < T_R\}} - B_{t,n} \mathbf{1}_{\{t \geq T_R\}} + R_{t,n}^{surv} - q_{y+t} Beq_{t,n}, \quad (2)$$

$$t \in \{t_0, \dots, T-1\}, n \in \mathcal{N}_t,$$

$$X_{i,t,n} = X_{i,t,n}^{\rightarrow} + X_{i,t,n}^{buy} - X_{i,t,n}^{sell}, \quad t \in \{t_0, \dots, T-1\}, n \in \mathcal{N}_t, i \neq 1, \quad (3)$$

$$X_{i,t,n}^{\rightarrow} = (1 + r_{i,t,n}) X_{i,t-,n-}, \quad t \in \{t_1, \dots, T\}, n \in \mathcal{N}_t, i \in \mathcal{I}, \quad (4)$$

$$P_{t,n}^{tot} \leq (p^{vol} + p^{fixed}) l_t, \quad t \in \{t_0, \dots, T-1\}, n \in \mathcal{N}_t, \quad (5)$$

$$B_{t,n}^{tot} = B_{t,n} + b_t^{state}, \quad t \in \{t_0, \dots, T-1\}, n \in \mathcal{N}_t, \quad (6)$$

$$B_{t,n}^{tot} \geq b_t^{min}, \quad t \in \{t_0, \dots, T-1\}, n \in \mathcal{N}_t, \quad (7)$$

$$\sum_i X_{i,T,n}^{\rightarrow} \geq x_T^{min}, \quad n \in \mathcal{N}_T, \quad (8)$$

$$Beq_{t,n} = inst \sum_i X_{i,t,n}^{\rightarrow}, \quad t \in \{t_0, \dots, T-1\}, n \in \mathcal{N}_t, \quad (9)$$

$$R_{t,n}^{surv} = q_{y+t} \sum_i X_{i,t,n}^{\rightarrow}, \quad t \in \{t_0, \dots, T-1\}, n \in \mathcal{N}_t, \quad (10)$$

$$X_{i,t,n} \leq u_i \sum_i X_{i,t,n}, \quad t \in \{t_0, \dots, T-1\}, n \in \mathcal{N}_t, i \in \mathcal{I}, \quad (11)$$

$$X_{i,t,n} \geq d_i \sum_i X_{i,t,n}, \quad t \in \{t_0, \dots, T-1\}, n \in \mathcal{N}_t, i \in \mathcal{I}, \quad (12)$$

$$X_{i,t,n}^{buy} \geq 0, X_{i,t,n}^{sell} \geq 0, B_{t,n} \geq 0, Beq_{t,n} \geq 0, \quad t \in \{t_0, \dots, T-1\}, n \in \mathcal{N}_t, i \in \mathcal{I}. \quad (13)$$

Expression $\mathbf{1}_{\{(\cdot)=t\}}$ denotes an indicator function equal to 1 if $(\cdot) = t$ and 0 otherwise.

4 Intuition behind the optimal policy

Stochastic programming approach has highly practical application, and in contrast to stochastic optimal control, can handle more realistic constraints. However, because MSP calculates the optimal decisions numerically at each node of the scenario tree, it may be difficult to interpret the results. Accordingly, to understand the optimal solution, we take a closer look at the explicit formulae obtained for a simplified model via SOC approach.

Because the explicit solution for the case with two sources of retirement income (state retirement pension and benefits from the DC plan) has not been presented in the literature, we derive the optimal decisions in B. Nevertheless, to be able to derive the explicit solutions we have to simplify the model by introducing the following assumptions: i) a continuous-time setting, ii) no upper or lower bounds on the variables (such as those in equations (7)-(8) and (11)-(13)), iii) a risk-free return on cash, and iv) either a deterministic or optimal death benefit. Otherwise, obtaining the explicit solution is non-trivial.

Optimal investment The optimal investment decision for the presented model is of the form obtained by (Richard, 1975). Specifically, eq. (26)-(27) indicate that the proportion invested in the risky portfolio (i.e. portfolio consisting only of the risky assets) depends on the risk aversion, the market parameters, the value of the savings and the present value of the expected retirement state pension, g_t ; whereas the proportions between the assets in the risky portfolio depend on their expected returns, volatilities, and the correlations between the assets. If the individual expects no retirement state pension, the optimal strategy suggests a fixed-mix portfolio, as suggested by (Merton, 1969, 1971). Otherwise the individual should decrease the percentage in the risky portfolio as g_t decreases.

Optimal annuity payments Not less important is to determine the optimal annuity payments. Is there a withdrawal rate, according to which the accumulated savings should be spent, as inves-

tigated in e.g. (Bengen, 1994) and (Horneff et al., 2008)?

To understand formula (23), let us focus on the individual upon retirement (i.e. 65-year old), and let assume the risk-free investment, so we can separate the annuity payments decision from the investment decision. Given that the subjective mortality rate is equal to the pricing mortality rate, $\mu_t = \nu_t$, the payout curve is: constant if the impatience factor is equal to the risk-free rate, $\rho = r$, decreasing for impatient individuals, $\rho > r$, and increasing for patient ones, $\rho < r$. The parameter γ controls the slope of the payout curve. For the less risk averse individuals, such as $\gamma = -2$, the difference between the benefits received at the beginning and at the end of retirement is bigger than for moderately risk averse persons with $\gamma = -4$. The optimal payout profile for different choices of γ and ρ is illustrated on Fig. 3a. None of these payout profiles is better than the other; they are all optimal for individuals with different preferences.

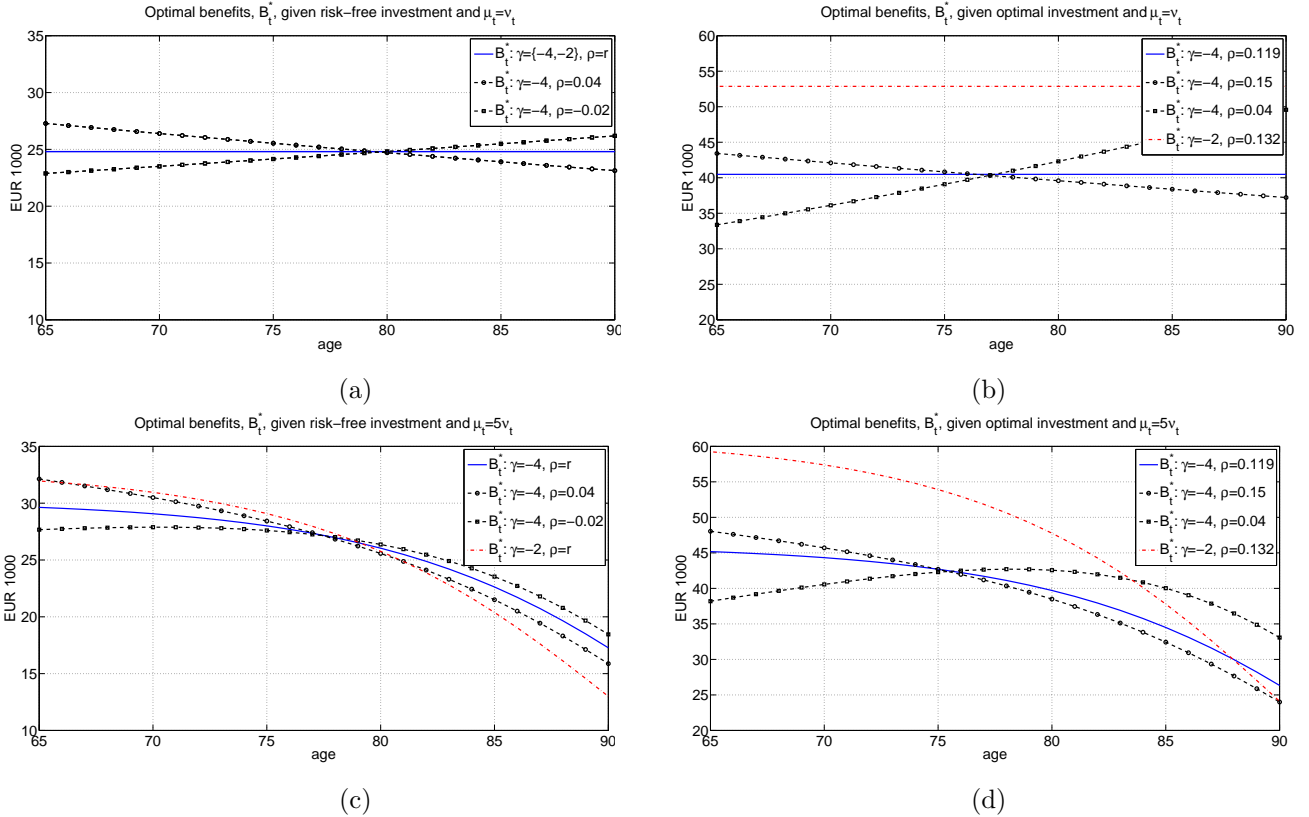


Figure 3: The optimal annuity payments for a given investment strategy and parameters γ , ρ and μ_t : Figures (a) and (c) assume the risk-free investment, figures (b) and (d) assume the optimal investment: {cash, bonds, dom. stocks, int. stocks}={10.7%, 49.0%, 27.9%, 12.3%} for $\gamma = -4$ and {cash, bonds, dom. stocks, int. stocks}={-48.8%, 81.7%, 46.6%, 20.5%} for $\gamma = -2$. Furthermore, Figures (c) and (d) assume the expected lifetime shorter than for an average individual, $\mu_t = 5\nu_t$. Parameters: $age_0 = 65$, $x_0 = 650$, $k = 3125$, and $b_t^{state} = 0$. The amounts are in EUR 1,000.

Investing in risky assets directly affects the size of annuity payments. To avoid the danger of running out of the resources before the individual's death, the benefits must be adjusted each year to account for the capital gains and losses. Nevertheless, despite these adjustments, we can

Age	MSP					
	65	70	75	80	85	90
<i>risk-free investment, moderate risk aversion ($\gamma = -4$)</i>						
constant benefits, $\rho = r$	3.8%	4.4%	5.3%	6.4%	7.8%	9.6%
decreasing benefits, $\rho = 0.04$	4.2	4.8	5.6	6.7	8.1	9.9
increasing benefits, $\rho = -0.02$	3.5	4.1	5.0	6.1	7.5	9.3
shorter expected lifetime, $\mu_t = 5\nu_t$, $\rho = r$	4.6	5.5	6.7	8.3	10.5	13.0
<i>optimal investment, moderate risk aversion ($\gamma = -4$), i.e., {cash, bonds, dom. stocks, int. stocks}={10.7%, 49.0%, 27.9%, 12.3%}</i>						
constant benefits, $\rho = 0.119$	6.2	6.8	7.5	8.5	9.8	11.4
decreasing benefits, $\rho = 0.15$	6.7	7.2	8.0	8.9	10.2	11.7
increasing benefits, $\rho = 0.04$	5.1	5.7	6.5	7.6	8.9	10.6
shorter expected lifetime, $\mu_t = 5\nu_t$, $\rho = 0.119$	7.0	7.8	8.9	10.5	12.5	14.8
<i>optimal investment, lower risk aversion ($\gamma = -2$), i.e., {cash, bonds, dom. stocks, int. stocks}={-48.8%, 81.7%, 46.6%, 20.5%}</i>						
constant benefits, $\rho = 0.132$	8.1	8.6	9.3	10.2	11.3	12.7

Table 1: Optimal withdrawal rates for a given investment strategy and parameters γ , ρ and μ_t . Parameters: $age_0 = 65$, $x_0 = 650$, $k = 3125$, and $b_t^{state} = 0$.

still control the expected payout curve; we can choose parameters (γ, ρ, μ_t) such that the expected payout curve is constant, increasing or decreasing. Given the optimal investment strategy, eq. (26), we obtain constant expected annuity payments for $\rho = r + (2 - \gamma) \frac{(\alpha - r)^2}{2\sigma^2(1 - \gamma)}$ and $\mu_t = \nu_t$. Any other choice of ρ and μ_t leads to either increasing or decreasing payout curve, as shown on Figs. 3b and 3d.

Following this argumentation, one can recognize that formula (23) defines equity-linked annuity payments. The more aggressive investment strategy, the higher expected benefits. A person interested in a constant payout curve would expect: EUR 52,900 given $\gamma = -2$ and $\rho = 13.2\%$, EUR 40,500 given $\gamma = -4$ and $\rho = 11.9\%$, and EUR 24,800 given the risk-free investment, $\rho = r$ and any choice of γ . This formula also shows, that indeed there exists an optimal withdrawal rate $1/\bar{a}_t^*$ that depends on the constants γ and ρ characterizing the individual's risk tolerance and impatience, and on the subjective mortality rate μ_t . Interestingly, the withdrawal rate is not only a fraction of the savings at a given time, but also of the present value of the expected retirement state pension. Accordingly, the size of the benefits expected from the state retirement pension affects the optimal size of the payments from the DC plan. The optimal withdrawal rates for different values of γ , ρ and μ_t are presented in Table 1.

Finally, Figs. 3c-3d show how the subjective lifetime expectancy affects the optimal payout curve. The choice of $\mu_t = 5\nu_t$ indicates that the individual expects to die earlier than an average individual assumed by the life insurer. Specifically, for the chosen mortality model, such a choice of μ_t corresponds to the expected lifetime of 78.7 years, with 70.2% chances to survive until age 75 and only 18.5% chances to survive until age 85; (given that the individual is alive at age 65). Independently of the choice of γ and ρ , the payout curve is no longer constant, but decreases proportionally

to the probability of survival. This result indicates that a life annuity with a decreasing payout curve is preferable than for example a term annuity, which pays constant the benefits for 10 or 25 years. A similar conclusion has been drawn by (Milevsky and Huang, 2011), who argue that "the optimal ... behavior in the face of personal longevity risk is to consume in proportion to survival probabilities - adjusted upward for pension income and downward for longevity risk aversion - as opposed to blindly withdrawing constant income for life".

Optimal death sum The optimal death sum, eq. (24), is a linear function of the optimal annuity payments. Both decisions are proportional by the factor $\left(k \frac{\mu_t}{\nu_t}\right)^{1/(1-\gamma)}$, which changes with the strength of the bequest motive k , risk aversion, and the relation between the subjective and pricing mortality rates. Therefore, similarly as expected annuity payments, the expected death sum can be constant, increasing or decreasing, whereas the actual size of the death sum depends on the realized portfolio returns. Formula (24) defines moreover the optimal death sum rate as a proportion $\frac{1}{\bar{a}_t^*} \left(k \frac{\mu_t}{\nu_t}\right)^{1/(1-\gamma)}$ of the current savings and the present value of expected state retirement pension.

5 Numerical results

To present the application of the model we have chosen a number of individuals with different economical and personal characteristics. Even small-scale optimization problems, such as problems based on 1,250 scenarios (4 periods with branching factors $\{10,5,5,5\}$), are sufficient to present the applications of the model. The MSP formulation can be implemented on a personal computer and takes only a few seconds to run. We implemented the program on a Dell computer with an Intel Core i5-2520M 2.50 GHz processor and 4 GB RAM, using Matlab 8.2.0.713 (R2013b), and GAMS 24.1.3 with non-linear solver MOSEK 7.0.0.75. The optimization module can also be solved with a linear or quadratic solver, such as CPLEX, but the objective function has to be linearly or quadratically approximated. Furthermore, to check the robustness of the results, we rerun the model for 30 different scenario trees. Thus, the results are based on $1,250 \cdot 30 = 62,500$ scenarios.

The numerical examples provide some guidelines to individuals in DC pension plans on how to manage their savings both before and after retirement. These guidelines can also be used by life insurers for designing pension products that are highly customised to the individuals' needs. When speaking of improving pension product design, we refer to pension savings management that combines three important decisions: investment, annuity payments and the level of death sum. These decisions are optimal for a particular individual, therefore, each person should have a different pension product.

Because most of the considered products allow for the investment in risky assets, guarantee payments as long as the person is alive, and pay out a death sum upon the individual's death, we call these products *optimal equity-linked life annuities*. Additionally, depending on the time of the

purchase of the product, we distinguish between *immediate* and *deferred* annuities. Both products start paying out the benefits upon retirement, but deferred annuity is purchased when the individual is still employed. Furthermore, during the deferment period the premiums are invested according to the optimal investment strategy, and the life insurance policy is effective.

5.1 Optimal immediate equity-linked life annuity

We start with a 65-year old female¹. She is just retiring and is interested in purchasing an immediate equity-linked life annuity. She has saved $x_0 = 650,000$ (EUR) on her pension account, has moderate risk aversion $1 - \gamma = 5$, and expects to live as assumed by the insurer ($\mu_t = \nu_t$), i.e. on average until age 89.1. During retirement she expects the benefits from the state, $b_t^{state} = 4,000$ (EUR). When asked about the preferable payout profile, she chooses life long increasing payments. Such a payout curve can be obtained for e.g. $\rho = 0.04$. She has a bequest motive but is not sure how much money to bequeath to her heirs, and how it will affect the level of the annuity payments. Therefore, we investigate three cases: no bequest motive, $k = 0$, the death sum equal to the level of savings, $ins_t = 1$, and the death sum equal to the sum of the benefits received over 5 years (obtained for $k = 5^{1-\gamma} \frac{\nu_t}{\mu_t} = 3125$). The life insurer does not allow for gearing or shorting the assets, thus $d_i = 0$ and $u_i = 1$.

Table 2, sections (a), (b) and (c), present the optimal decisions for a person with such characteristics. The first 10 years of the retirement are modeled using MSP approach with the intervals between the decisions of $\Delta t = \{1, 3, 3, 3\}$ years. Thereafter, we approximate the model with its simpler continuous-time version that can be solved explicitly using SOC approach. Reading Table 2, we can observe three important facts. First, the optimal investment strategy is almost identical for the considered different weights on the bequest motive. This result can be surprising at first, but eq. (26) states that the only parameters that influence the investment decisions, are the market parameters, the risk aversion, the current level of savings, and the present value of expected retirement state pension. For $\gamma = -4$ the majority of savings upon retirement is invested in bonds (53%), domestic stocks (32%), and international stocks (13%). These proportions change slowly so that the individual invests less in risky assets as the present value of expected state retirement benefits decreases.

Second, as chosen by the individual, the expected annuity payments increase. A person without a bequest motive would receive the highest payments, not only because she does not pay for the insurance coverage, but also because she receives a survival credit for each year she survives. Upon retirement our individual will obtain the total yearly benefits of EUR 42,000, 36,600 and 36,900, respectively for the cases without a bequest motive, with a death sum equal to the level of savings, and with an optimal death sum given $k = 3125$. Twenty years after retirement the payments are

¹We follow the approach common for the most European countries, where the price of annuities does not depend on the gender, i.e. even though women are expected to live longer, they are entitled to receive the same benefits as men.

Age	MSP				SOC		
	65	66	69	72	$age_T=75$	80	85
<i>(a) Without a bequest motive, $k = 0$.</i>							
Cash	1%	0%	0%	0%	1%	2%	3%
Bonds	53	56	57	58	54	54	53
Dom. stocks	32	30	30	29	31	31	31
Int. stocks	14	14	13	13	14	13	13
Total benefits, B_t^{tot*}	€42.0	€42.7	€44.7	€46.7	€48.5	€52.5	€56.8
Bequest amount, Beq_t^*	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Value of savings, X_t^*	650.0	645.4	616.6	577.2	525.8	457.4	377.8
<i>(b) With a death sum equal to value of savings, $ins_t = 1$.</i>							
Cash	2%	0%	0%	0%	2%	4%	5%
Bonds	53	57	57	58	54	53	52
Dom. stocks	32	30	30	29	31	30	30
Int. stocks	14	14	13	13	13	13	13
Total benefits, B_t^{tot*}	€36.6	€37.1	€38.8	€40.4	€42.1	€45.5	€49.3
Bequest amount, Beq_t^*	650.0	648.7	636.8	616.4	210.3	227.6	246.4
Value of savings, X_t^*	650.0	648.7	636.8	616.4	586.1	556.0	518.7
<i>(c) With an optimal death sum given $k = 3125$.</i>							
Cash	2%	0%	0%	0%	2%	4%	5%
Bonds	53	57	57	58	54	53	52
Dom. stocks	32	30	30	29	31	30	30
Int. stocks	13	13	13	13	13	13	13
Total benefits, B_t^{tot*}	€36.9	€37.5	€39.2	€40.9	€42.6	€46.1	€49.9
Bequest amount, Beq_t^*	184.4	187.3	195.9	204.7	212.9	230.5	249.5
Value of savings, X_t^*	650.0	650.2	639.4	621.1	594.2	563.6	525.6
<i>(d) With a minimum level of benefits, $b_t^{min} = 28$, minimum level of savings upon horizon, $x_T = 370$, and optimal death sum given $k = 3125$.</i>							
Cash	27%	1%	1%	2%	2%	4%	5%
Bonds	43	61	60	59	54	53	52
Dom. stocks	22	26	28	28	31	30	30
Int. stocks	8	11	11	12	13	13	13
Total benefits, B_t^{tot*}	€34.9	€35.9	€37.9	€40.0	€41.9	€45.3	€49.1
Bequest amount, Beq_t^*	174.6	179.1	189.0	198.5	209.5	226.7	245.4
Value of savings, X_t^*	650.0	644.0	631.0	611.4	583.6	553.7	516.6
<i>(e) With shorter expected lifetime than an average individual, $\mu_t = 5\nu_t$, and optimal death sum given $k = 3125$.</i>							
Cash	2%	0%	0%	0%	1%	2%	2%
Bonds	53	56	57	58	54	54	54
Dom. stocks	32	30	30	29	31	31	31
Int. stocks	13	14	13	13	14	13	13
Total benefits, B_t^{tot*}	€42.2	€42.8	€44.2	€45.4	€46.0	€46.3	€43.6
Bequest amount, Beq_t^*	290.3	294.1	303.8	311.5	317.0	319.1	300.1
Value of savings, X_t^*	650.0	644.1	613.5	572.9	521.7	441.3	343.0

Table 2: The optimal asset allocation, total benefits and size of death sum for a 65-year old individual given: (a) no bequest motive, (b) a death sum equal to the value of savings, (c) an optimal death sum given $k = 3125$, (d) a minimum level of benefits $b_t^{min} = 28$ and a minimum level of savings upon horizon $x_T^{min} = 370$, and (e) a subjective lifetime expectancy $\mu_t = 5\nu_t$. The numbers are presented in terms of means across the nodes associated with each period and the scenario trees. Parameters: $age_0 = 65$, $x_0 = 650$, $b_t^{state} = 4$, $\gamma = -4$, $\rho = 0.04$, $d_i = 0$, $u_i = 1$, $\mu_t = \nu_t$, $T = 10$, and $\Delta t = \{1, 3, 3, 3\}$. The asset allocations are in percentages and other amounts are in EUR 1,000.

expected to increase to EUR 56,800, 49,300 and 49,900, respectively.

Third, looking closer at the case with the optimal death sum, we find that the death benefit increases with time, and that it is much lower than the value of savings. As explained in Sec. 4, the optimal death sum is proportional to the annuity payments by a factor $\left(k \frac{\mu_t}{\nu_t}\right)^{1/(1-\gamma)} = 5$. Therefore, increasing annuity payments imply increasing death benefit. Furthermore, Beq_t^* is much lower than the value of savings for most of the retirement; it is higher than the value of savings only during the very late years (e.g. later than age 95), when the individual has already spent most of her savings.

Recall that Table 2 presents the means across the nodes assigned to each time period, across the scenarios, and across different scenario trees. The optimal values are stochastic and differ for each realization of the asset returns. Equity-linked payments may vary significantly, especially after a longer period such as 10 years. Even though the expected benefits are increasing, if for 10 years in a row the risky assets bring losses, our individual may end up only with yearly payments of EUR 20,000 at age 75. See Fig. 4a. To mitigate this risk, we can add a lower limit on the benefits' size by adding constraints (7) and (8) in the MSP formulation. The results in Table 2, section (d) and Fig. 4b, show that these constraints affect the optimal decisions. The asset allocation is more conservative during the first years of retirement, and leads to on average lower annuity payments and death sum. Other studies have shown that adding the guarantees to pension products increases the life insurer's liabilities, and thus prevents them from offering greater investment opportunities, see e.g. (Guillén et al., 2013). In this example, a guarantee that the minimum payment never falls below $b_t^{min} = 28,000$ (EUR) is only added for the first 10 years after retirement. Nevertheless, such a guarantee reduces the expected yearly benefits from EUR 36,900 and 49,900 (upon ages 65 and 85) to EUR 34,900 and 49,100, respectively.

Finally, what can we recommend if our individual has a bad health condition and expects to die earlier than an average individual? To illustrate such a case, we choose $\mu_t = 5\nu_t$, that is, the expected lifetime of our individual is 78.7 years, which is approximately 10 years shorter than what the insurer assumes. Given that as in many European countries the survival credit and the price for life insurance are calculated under unisex criteria and are not subject to health screening, the individual should spend more savings during the first years of the retirement. The optimal solution (Table 2, section (e)) clearly suggests to change the payout curve so that the expected benefits decrease proportionally to the probability of survival, ${}_t\tilde{p}_y$, as well as to increase the death sum. The optimal investment strategy remains similar as in the case with the average lifetime expectancy.

5.2 Optimal deferred equity-linked life annuity

This section focuses on the decisions that an individual faces during the accumulation phase (i.e. before retirement). Our person is a 45-year old female with initial savings of $x_0 = 130,000$ (EUR) and pension contributions of 10% of her salary. The yearly salary, $l_t = 50,000$ (EUR), increases

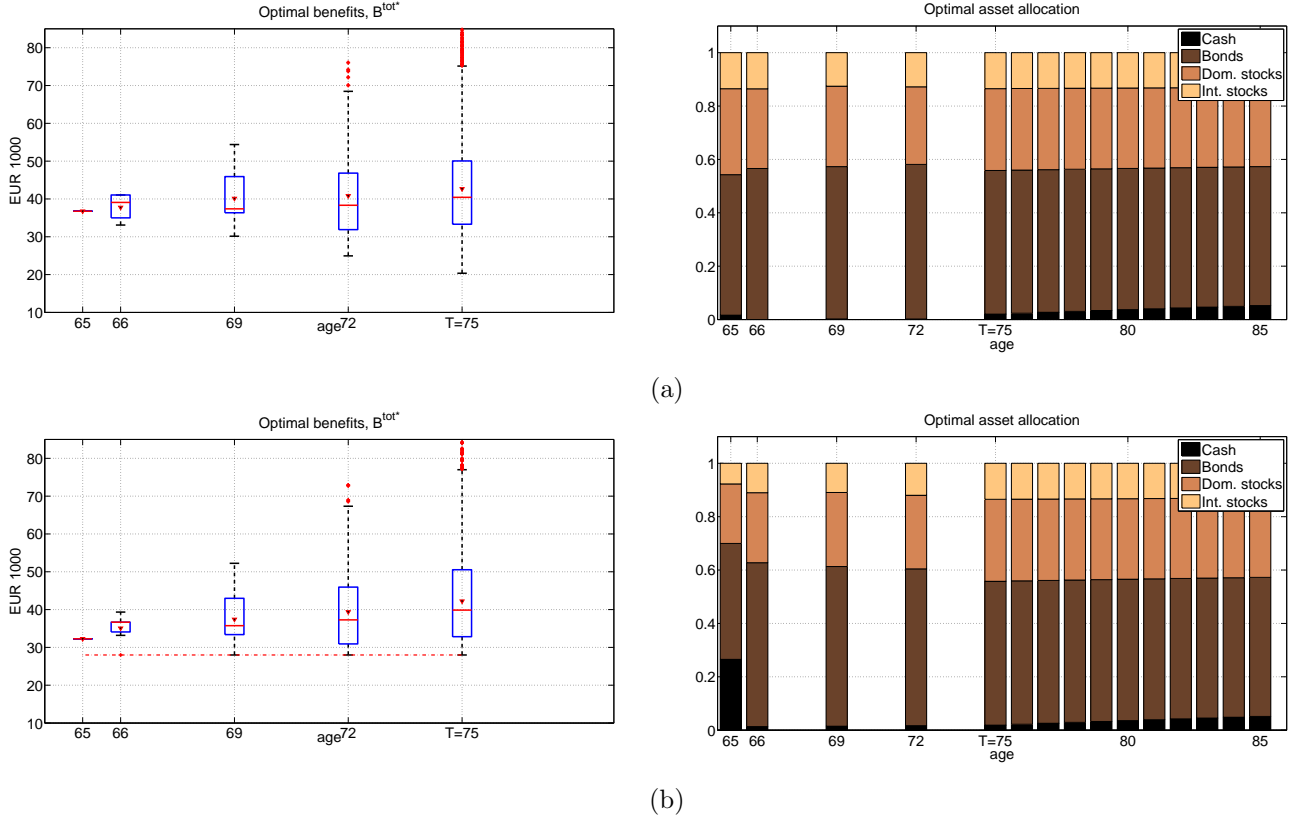


Figure 4: The probability distribution of the optimal total benefits in one scenario tree, $B_t^{tot*} = B_t^* + b_t^{state}$, and the optimal investment in terms of means across the nodes and the scenario trees given: (a) an optimal death sum given $k = 3125$, and (b) an optimal death sum given $k = 3125$, the minimum level of benefits $b_t^{min} = 28$, and the minimum level of savings upon horizon, $x_T^{min} = 370$. The central mark in each box is the median, the triangle marker denotes the mean, the edges of the box are the 25th and 75th percentiles, the whiskers extend to the most extreme data points not considered outliers, and outliers are plotted individually with the red crosses. The red dashed line denotes the minimum level of benefits b_t^{min} . Parameters: $age_0 = 65$, $x_0 = 650$, $b_t^{state} = 4$, $\gamma = -4$, $\rho = 0.04$, $\mu_t = \nu_t$, $d_i = 0$, $u_i = 1$, $T = 10$, and $\Delta t = \{1, 3, 3, 3\}$. All the amounts are in EUR 1,000.

every year with $y_l = 2\%$. Having an average lifetime expectancy and anticipating $b_t^{state} = 4,000$ (EUR) from the retirement state pension, she would like to purchase an annuity that starts constant payments in 20 years (upon her retirement). She describes herself as moderate risk averse (e.g. $\gamma = -4$), therefore would like to invest some of her savings in risky assets. She has no further preferences on the portfolio composition but she faces short sales constraints on all assets.

The optimal decisions in this example are the investment strategy before and after retirement and the annuity payments after retirement. We also investigate the cases with and without a bequest motive. We divide the period of 20 years into 4 periods of 5 years each - we make the decisions every fifth year. The solution after retirement is calculated analytically using Hamilton-Jacobi-Bellman techniques for the simplified model.

Table 3 shows that, similarly as in the previous case, the bequest motive has a minor effect on the optimal asset allocation. The overall investment strategy suggests to decrease the risk as the

individual ages. The optimal portfolio consisting of 0% in cash, 33% in bonds, 46% in domestic stocks, and 21% in international stocks, (upon age 45), smoothly changes to {1%, 54%, 31%, 14%}, respectively, upon age 75. In the MSP formulation we assume that cash has a volatility of 3.8% and that no shorting or gearing is allowed, whereas for the period $[T, \tilde{T})$, which is solved using SOC approach, the model is simplified: cash is assumed to be risk-free and no constraints on the portfolio allocation are imposed. (Otherwise finding the analytical solution is non-trivial.) This difference in the assumptions cause the fluctuations in cash holdings between the periods covered by two different optimization approaches.

The bequest motive affects the level of annuity payments, but not the payout curve: the expected benefits are constant. A person without a bequest motive will receive the highest benefits, $E[B_t^{tot*}] = 50,100$ (EUR) per year, a person with a death sum equal to the value of savings ($ins_t = 1$) will receive payments of EUR 44,200 per year, and a person with an optimal death sum given parameter $k = 5^{1-\gamma} = 3125$ will receive payments of EUR 44,240 per year. Notice how small the difference between the annuity payments in the last two cases is. Because the probability that a 45-year old person survives until age 65 is high, the price for the life insurance is low. Therefore, the value of savings upon retirement X_T^* is similar in both cases, and implies the annuity payments of approximately the same level. After retirement the optimal death sum is constant and proportional to the annuity payments by factor 5.

Nevertheless, the numbers in Table 3 are the means across the scenarios, whereas their actual values depend on the realizations of the asset returns. Figure 5a (left) shows the probability distribution of the savings upon retirement for the case with the optimal death sum (for one scenario tree). After contributing to the pension account for 20 years and allocating the portfolio according to the optimal investment strategy, the individual should expect to save up EUR 622,400 upon retirement. This amount gives the expected total benefits of EUR 44,240, which is 60% of the individual's salary level upon retirement.

However, this amount can be much lower: in a scenario with long periods of negative returns, the person may end up with only EUR 200,000 on her savings account, which would provide the yearly retirement income of EUR 17,900. Thus, she may choose to increase the premiums by additional 5% and add a lower limit on the size of the savings upon retirement, for example $x_T^{min} > 300,000$ (EUR). This limit corresponds to the minimum level of benefits $b_t^{min} = 24,200$ (EUR). As illustrated in Table 3 and Fig 5b, both the probability distribution of savings and the optimal asset allocation change. The probability distribution has shifted to the right and the investment strategy implies slightly more conservative portfolio.

Is it possible to choose a higher limit x_T^{min} solely by adjusting the investment strategy? The answer depends on the available assets and their returns' distribution. To be certain that the value of savings will not fall below a pre-specified limit, we must employ a more conservative investment strategy. However, if the strategy is too conservative, it may not be possible to reach this level. Therefore, choosing too high values for x_T^{min} often leads to infeasible solution.

Age	MSP					SOC	
	45	50	55	60	$age_T=65$	70	75
<i>(a) Without a bequest motive, $k = 0$.</i>							
Cash	0%	0%	0%	0%	-2%	-1%	0%
Bonds	33	46	51	56	56	55	55
Dom. stocks	46	38	32	30	32	32	31
Int. stocks	21	16	17	14	14	14	14
Total benefits, B_t^{tot*}	€0.0	€0.0	€0.0	€0.0	€50.1	€50.1	€50.1
Bequest amount, Beq_t^*	0.0	0.0	0.0	0.0	0.0	0.0	0.0
Value of savings, X_t^*	130.0	211.7	316.7	451.6	623.6	553.2	473.3
<i>(b) With a death sum equal to value of savings, $ins_t = 1$.</i>							
Cash	0%	0%	0%	0%	-2%	0%	1%
Bonds	33	46	51	56	56	55	54
Dom. stocks	46	38	32	30	32	31	31
Int. stocks	21	16	17	14	14	14	14
Total benefits, B_t^{tot*}	€0.0	€0.0	€0.0	€0.0	€44.20	€44.20	€44.20
Bequest amount, Beq_t^*	130.0	211.7	316.5	451.0	221.0	221.0	221.0
Value of savings, X_t^*	130.0	211.7	316.5	451.0	621.8	578.3	528.0
<i>(c) With an optimal death sum given $k = 3125$.</i>							
Cash	0%	0%	0%	0%	-2%	0%	1%
Bonds	33	46	51	56	56	55	54
Dom. stocks	46	38	32	30	32	31	31
Int. stocks	21	16	17	14	14	14	14
Total benefits, B_t^{tot*}	€0.0	€0.0	€0.0	€0.0	€44.24	€44.24	€44.24
Bequest amount, Beq_t^*	233.0	227.8	224.4	222.2	221.2	221.2	221.2
Value of savings, X_t^*	130.0	211.7	316.4	451.0	622.4	578.8	528.4
<i>(d) With minimum level of savings upon retirement, $x_T^{min} = 300$, additional contributions $p^{vol} = 5\%$, and optimal death sum given $k = 3125$.</i>							
Cash	1%	0%	0%	1%	0%	1%	2%
Bonds	37	47	52	57	55	54	54
Dom. stocks	43	38	32	29	31	31	31
Int. stocks	19	15	16	13	14	14	13
Total benefits, B_t^{tot*}	€0.0	€0.0	€0.0	€0.0	€50.5	€50.5	€50.5
Bequest amount, Beq_t^*	262.8	258.5	254.9	252.8	252.5	252.5	252.5
Value of savings, X_t^*	130.0	225.6	351.5	514.6	723.0	671.0	611.5
<i>(e) With shorter expected lifetime than an average individual, $\mu_t = 5\nu_t$ and optimal death sum given $k = 3125$.</i>							
Cash	0%	0%	0%	0%	-2%	-1%	0%
Bonds	33	46	51	56	56	55	55
Dom. stocks	46	38	32	30	32	31	31
Int. stocks	21	16	17	14	14	14	14
Total benefits, B_t^{tot*}	€0.0	€0.0	€0.0	€0.0	€49.3	€48.4	€46.6
Bequest amount, Beq_t^*	364.1	355.6	349.3	344.2	339.5	333.1	320.9
Value of savings, X_t^*	130.0	211.6	316.3	450.7	621.7	548.2	463.1

Table 3: The optimal asset allocation, total benefits and size of death sum for a 45-year old individual given: (a) no bequest motive, (b) a death sum equal to the value of savings, (c) an optimal death sum given $k = 3125$, (d) a minimum level of savings upon horizon $x_T^{min} = 300$ and additional contributions $p^{vol} = 5\%$, and (e) a subjective lifetime expectancy $\mu_t = 5\nu_t$. The numbers are presented in terms of means across the nodes associated with each period and the scenario trees. Parameters: $age_0 = 45$, $x_0 = 130$, $l_0 = 50$, $y_t = 2\%$, $p^{fixed} = 10\%$, $p^{vol} = 0$, $b_t^{state} = 4$, $\gamma = -4$, $\rho = 0.119$, $d_i = 0$, $u_i = 1$, $\mu_t = \nu_t$, $T = 20$, and $\Delta t = \{5, 5, 5, 5\}$. The asset allocations are in percentages and other amounts are in EUR 1,000.

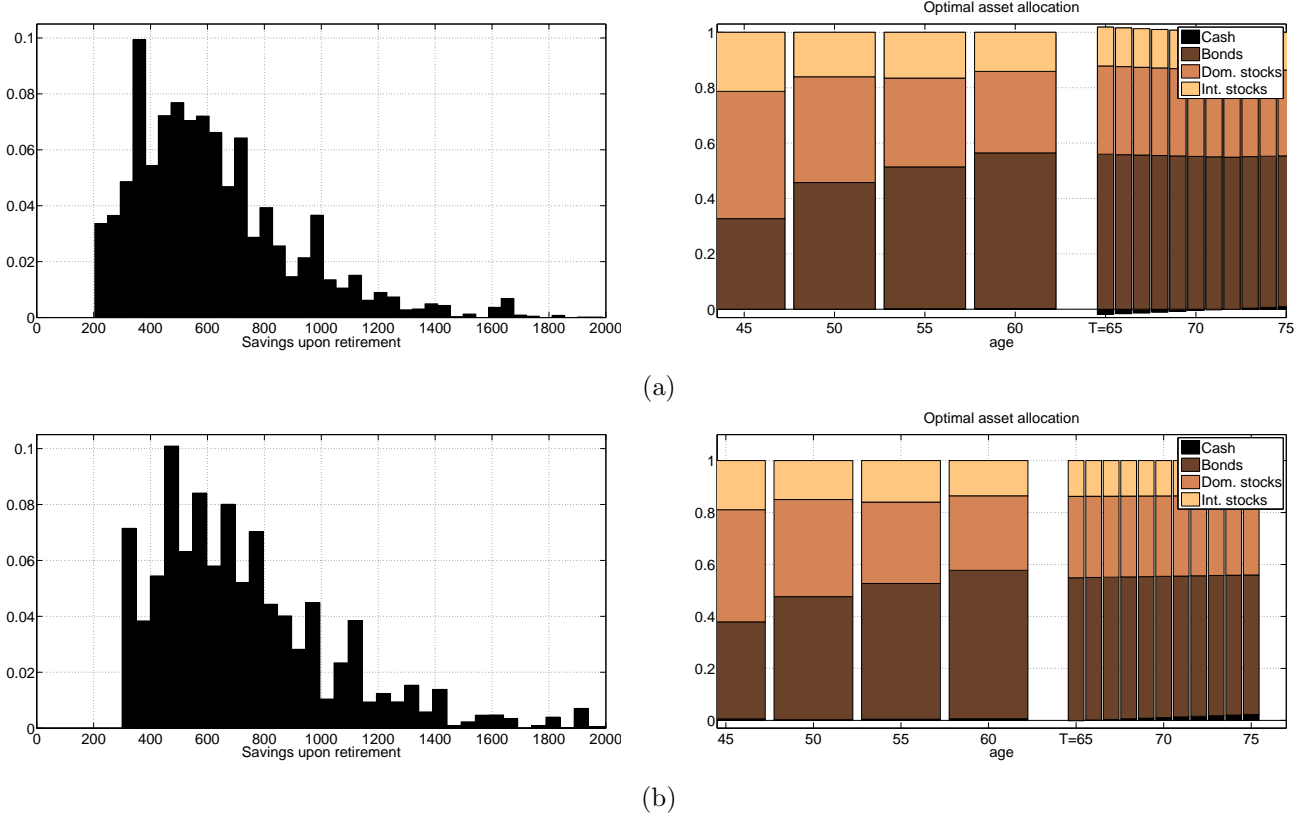


Figure 5: The probability distribution of the value of savings upon retirement X_T^* for one scenario tree, and the optimal investment in terms of means across the nodes and the scenario trees given: (a) an optimal death sum given $k = 3125$, and (b) an optimal death sum given $k = 3125$, the additional contributions $p^{vol} = 5\%$, and the minimum level of savings upon horizon, $x_T^{min} = 300$. Parameters: $age_0 = 45$, $x_0 = 130$, $l_0 = 50$, $y_l = 2\%$, $p^{fixed} = 10\%$, $b_i^{state} = 4$, $\gamma = -4$, $\rho = 0.119$, $\mu_t = \nu_t$, $d_i = 0$, $u_i = 1$, $T = 20$, and $\Delta t = \{5, 5, 5, 5\}$. The asset allocations are in percentages and other amounts are in EUR 1,000.

Finally, are optimal deferred life annuities still attractive if one expects to die earlier than an average person? We investigate the case for $\mu_t = 5\nu_t$, i.e. the expected lifetime of the individual is 78.7 years and the probability that she survives until age 85 is only 18%. A closer look at the optimal decisions (Table 3, section (e)) reveals that it is optimal for the person to invest in deferred life annuities only if the payout curve is decreasing. The optimal withdrawal rate is proportional to the probability of survival, therefore she should spend more savings in the beginning of retirement. The initial payment is EUR 5,000 higher than in the case with the average lifetime expectancy (compare with Table 3, section (c)), and the optimal death benefit increases significantly and stays above EUR 300,000 until age 72.

6 Conclusions and future work

This paper provides some guidelines to individuals with defined contribution pension plans. We argue that the decisions regarding the asset allocation, the annuity payments, and the size of

death sum should be highly customized. With several numerical examples we have illustrated how the optimal decisions depend on: 1) economical characteristics - such as current value on the pension savings account, expected pension contributions (mandatory and voluntary), and expected income after retirement (e.g. retirement state pension), and 2) personal characteristics - such as risk aversion, lifetime expectancy, preferable payout profile, bequest motive, and preferences on portfolio composition.

To help individuals manage their pension savings, we have built a model that combines two optimization techniques: multi-stage stochastic programming and stochastic optimal control. MSP especially has highly practical applications and generates results that are not only consistent with common knowledge about life-cycle asset allocation, but are also realistic. The presented model is flexible and can be applied either by financial advisers in countries where individuals have lots of flexibility in managing their pension savings, or by life insurers in countries where individuals are less involved in the savings and investment decisions. Because the operations research methods are not common in the actuarial literature, we argue that the presented optimization approach has potential to stimulate new thinking and add to actuarial practise.

This work could be improved in various ways. Investigation of the impact of the administration costs, transaction costs and taxes is definitely relevant from a practical point of view. Taxes are especially important, since in many countries life annuities are tax deferred investment vehicles, and therefore preferred to personal investment. Furthermore, one could incorporate in the model other sources of uncertainty such as stochastic longevity risk and uncertain salary progression.

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Appendices

A A short introduction to multi-stage stochastic programming

This appendix briefly introduces a multi-stage program with recourse. For a more detailed theory, see e.g. (Birge and Louveaux, 1997), (Zenios, 2008) and (Shapiro et al., 2009).

To start with, let us formulate a two-stage version of the problem with recourse. We keep the notation from the aforementioned books, and define $(\Omega, \mathcal{F}, \mathbb{P})$ to be a probability space, ω is an element (outcome) of a sample space Ω , and $\xi = \xi(\omega)$ is a random vector which belongs to the probability space with support $\Xi = \{\xi \in \mathbb{R}^N \mid 0 \leq \xi < \infty\}$. We need two vectors for decision variables to distinguish between the anticipative and adaptive policy:

- $y_0 \in \mathbb{R}^{n_0}$ - a vector of first-stage decisions, which are made before the random variables are observed; the decisions do not depend on the future observations but anticipate possible future realizations of the random vector,
- $y_1(\xi) \in \mathbb{R}^{n_1}$ - a random vector of second-stage decisions which are made after the random variables have been observed. They are constrained by decisions y_0 and depend on the realizations of the random vector ξ .

Once a first-stage decision y_0 has been made, some realization of the random vector can be observed. Then, the second-stage problem seeks a decision vector $y_1(\xi)$ that optimizes the function $f_1(y_1(\xi); \xi)$ for a given value of the first-stage decision y_0 and the random parameters $\{T_{0,1}(\xi), W_1(\xi), h_1(\xi) \mid \xi \in$

$\Xi\}$. Combining both stages, we obtain the following model:

$$\begin{aligned}
 \max_{y_0} \quad & f_0(y_0) + E \left[\max_{y_1} f_1(y_1(\xi); \xi) \mid \mathcal{F} \right], \\
 \text{s.t.} \quad & W_0 y_0 = h_0, \\
 & T_{0,1}(\xi) y_0 + W_1(\xi) y_1(\xi) = h_1(\xi), \\
 & y_0 \geq 0, \ y_1(\xi) \geq 0.
 \end{aligned} \tag{14}$$

The recourse problem can be extended to a multi-period stochastic programs, where observations and decisions are made at T different stages, which correspond to time instances when some information is revealed and a decision can be made. Let the random variable ξ have support $\Xi_1 \times \Xi_2 \times \dots \times \Xi_T$ and the observations are captured in the information sets $\{\mathcal{F}_t\}_{t=1}^T$ with $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \dots \subset \mathcal{F}_T$. For each stage $t = 1, \dots, T$, $y_t(\omega) \in \mathbb{R}^{n_t}$ denotes the recourse decision variable vector optimizing the random objective function $f_t(y_t(\xi_t); \xi_t)$, given the random parameters $\{T_{t-1,t}(\omega_t), W_t(\omega_t), h_t(\omega_t) \mid \xi_t \in \Xi_t\}$. Then, the following actions are taken at each stage:

decision $y_0 \rightarrow$ observation $\xi_1 := (T_{0,1}, W_1, h_1) \rightarrow$ decision $y_1 \rightarrow \dots$
 \rightarrow observation $\xi_T := (T_{T-1,T}, W_T, h_T) \rightarrow$ decision y_T ,

which can be formulated as the following multi-stage program:

$$\begin{aligned}
 \max_{y_0} \quad & f_0(y_0) + E \left[\max_{y_1} f_1(y_1; \xi_1) + \dots E \left[\max_{y_T} f_T(y_T; \xi_T) \mid \mathcal{F}_T \right] \dots \mid \mathcal{F}_1 \right], \\
 \text{s.t.} \quad & W_0 y_0 = h_0, \\
 & T_{t-1,t}(\xi_t) y_{t-1}(\xi_{t-1}) + W_t(\xi_t) y_t(\xi_t) = h_t(\xi_t), & t = 1, \dots, T, \\
 & y_0 \geq 0, \ y_t(\xi_t) \geq 0, & t = 1, \dots, T.
 \end{aligned} \tag{15}$$

By the tower property of conditional expectation we can rewrite the objective function of the above problem as:

$$\max_{y_0, y_1, \dots, y_T} f_0(y_0) + \sum_{t=1}^T E [f_t(y_t; \xi_t) \mid \mathcal{F}_1]. \tag{16}$$

Finally, if the random vector ξ_t has a discrete distribution with a finite number of possible realizations with the corresponding probabilities $prob_n$, the equation (16) can be rewritten as follows:

$$\max_{y_0, y_1, \dots, y_T} f_0(y_0) + \sum_{t=1}^T \sum_{n \in \mathcal{N}_t} f_t(y_t; \xi_t) \cdot prob_n. \tag{17}$$

B The end effect

The main drawback of multi-stage stochastic programs is the limited ability to handle many time periods under sufficient uncertainty. The scenario tree grows exponentially with each time period, therefore solving the problem becomes soon computationally intractable. To ensure, that the optimization problem covers the decisions for the entire lifetime of the individual, we incorporate the end effect in the objective function of the MSP formulation, eq. (1). The end effect is equal

to the optimal value function, which can be calculated explicitly using stochastic optimal control (Hamilton-Jacobi-Bellman techniques), and covers the remaining years, i.e. the interval $[T, \tilde{T}]$. However, to be able to derive the explicit solution, we have to simplify the model by introducing the following assumptions: i) a continuous-time setting, ii) no upper or lower bounds on the variables (such as those in equations (7)-(8) and (11)-(13)), iii) a risk-free return on cash, and iv) either a deterministic or optimal death benefit; as it has been done in the classical literature on optimal consumption and investment, (Merton, 1969, 1971) and life insurance, (Richard, 1975) and (Kraft and Steffensen, 2008).

Assume that the economy is represented by a standard Brownian motion W defined on the measurable space (Ω, \mathcal{F}) , where \mathcal{F} is the natural filtration of W . The space is equipped with the equivalent probability measures: objective measure \mathbb{P} and the martingale measure \mathbb{P}^* . The latter is used by the insurer to price the financial assets and life insurance, and to calculate the level of the benefits. The individual invests the proportion $1 - \Pi_t$ of her savings in a risk-free asset (cash) with a constant interest rate r and Π_t in a mutual fund consisting of $N - 1$ assets, which prices are log-normally distributed. Then, the mutual fund follows the dynamics $dS_t = \alpha S_t dt + \sigma S_t dW_t$, where

$$\alpha = \sum_{i=1}^{N-1} \theta_i \alpha_i, \quad \sigma^2 = \sum_{i=1}^{N-1} \sum_{j=1}^{N-1} \theta_i \theta_j \sigma_{ij}, \quad dW = \sum_{i=1}^{N-1} \theta_i \frac{\sigma_i}{\sigma} dW_i,$$

θ_i is the proportion of asset i in the mutual fund, and $\{\alpha_i, \sigma_{ij}\}$ define the physical distribution of the returns. The assets are correlated with the coefficient $corr_{ij}$, thus $\sigma_{ij} = \sigma_i \sigma_j corr_{ij}$.

The mortality rates μ_t and ν_t are assumed to be continuous and deterministic, and are defined by the jump intensities of the finite state Markov chain Z . Process Z is defined on the measurable space (Ω, \mathcal{F}) and is independent of the process W . We have calibrated ν_t to the Danish mortality rates and obtained a satisfactory curve fit for a function

$$\nu_t = a_1 \exp \left(- \left(\frac{t-b_1}{c_1} \right)^2 \right) + a_2 \exp \left(- \left(\frac{t-b_2}{c_2} \right)^2 \right), \quad (18)$$

where constants $a_1, b_1, c_1, a_2, b_2, c_2$ are defined in Table 4. Data, which include the mortality improvements, can be downloaded from Danish Financial Supervisory Authority website, see (Finanstilsynet, 2012). We further assume that the subjective mortality rate μ_t is proportional to ν_t .

During retirement, $T \geq T_R$, the dynamics of the savings account, while the person is alive, are given by

$$dX_t = (r + \Pi_t(\alpha - r))X_t dt + \Pi_t \sigma X_t dW_t - \nu_t Beq_t dt + \nu_t X_t dt - B_t dt, \quad (19)$$

$$X_0 = x_T,$$

where (X_t, B_t, Beq_t) are continuous-time variables corresponding to variables $(\sum_i X_{i,t,n}, B_{t,n}, Beq_{t,n})$ defined in the MSP formulation. Note that in the continuous-time framework we do not distinguish between the value of the savings before and after rebalancing. Moreover, rather than keeping the track of the traded amounts, we calculate the optimal asset allocation Π_t in the portfolio directly. The objective is to maximize the expected utility of total benefits and bequest, given that the

individual is alive at time t and has $X_t = x_t$ on her savings account:

$$\begin{aligned} V(t, x) &= \sup_{(\Pi_t, B_t, Beq_t) \in \mathcal{Q}[t, \tilde{T}]} E_{t,x} \left[\int_t^{\tilde{T}} s_{-t} \tilde{p}_{y+t} \left(u(s, b_s^{state} + B_s) + \mu_s k u(s, Beq_s) \right) ds \right], \\ V(\tilde{T}, x) &= 0. \end{aligned}$$

The expression $E_{t,x}$ denotes the conditional expectation under \mathbb{P} , whereas \mathcal{Q} is the set of control processes are admissible at time t . Both utilities are multiplied by the subjective probability that the individual survives until time $s > t$, given she has survived until time t . The utility of bequest is moreover multiplied by the probability of dying shortly after surviving until time s . Parameter k denotes the weight on the bequest motive relatively to the benefits, and \tilde{T} is a fixed time point at which the individual is dead with certainty.

This simplified problem can be solved explicitly using the Hamilton-Jacobi-Bellman techniques. In this appendix we derive the optimal value function and the optimal controls only for the period after retirement, $t > T_R$. The case for $t \leq T_R$, is slightly more complicated but can be derived in a similar way.

Based on the savings dynamics, eq. (19), the HJB equation for the considered period is defined as follows,

$$\begin{aligned} \frac{\partial V(t,x)}{\partial t} - \mu_t V + \sup_{(\Pi_t, B_t, Beq_t)} \left\{ \frac{1}{\gamma} w^{1-\gamma} (B_t + b_t^{state})^\gamma - B_t \frac{\partial V(t,x)}{\partial x} + \mu_t k \frac{1}{\gamma} w^{1-\gamma} Beq_t^\gamma - \nu_t Beq_t \frac{\partial V(t,x)}{\partial x} \right. \\ \left. + (r + \Pi_t(\alpha - r) + \nu_t) x \frac{\partial V(t,x)}{\partial x} + \frac{1}{2} \Pi_t^2 \sigma^2 x^2 \frac{\partial^2 V(t,x)}{\partial x^2} \right\} = 0, \\ V(\tilde{T}, x) = 0. \end{aligned}$$

We guess the solution

$$V(t, x) = \frac{1}{\gamma} f_t^{1-\gamma} (x + g_t)^\gamma \quad (20)$$

and verify that it is correct. By plugging in the derivatives to the HJB equation we find the functions g_t and f_t :

$$\begin{aligned} & \frac{1-\gamma}{\gamma} f_t^{-\gamma} \frac{\partial f_t}{\partial t} (x + g_t)^\gamma + f_t^{1-\gamma} (x + g_t)^{\gamma-1} \frac{\partial g_t}{\partial t} - \mu_t \frac{1}{\gamma} f_t^{1-\gamma} (x + g_t)^\gamma \\ & + \frac{1}{\gamma} w^{1-\gamma} \left(\frac{w_t}{f_t} (x + g_t) \right)^\gamma - \left(\frac{w_t}{f_t} (x + g_t) - b_t^{state} \right) f_t^{1-\gamma} (x + g_t)^{\gamma-1} \\ & + \mu_t \frac{1}{\gamma} k w^{1-\gamma} \left(k \frac{\mu_t}{\nu_t} \right)^{\gamma/(1-\gamma)} \frac{w_t^\gamma}{f_t^\gamma} (x + g_t)^\gamma - \nu_t \left(k \frac{\mu_t}{\nu_t} \right)^{1/(1-\gamma)} \frac{w_t}{f} (x + g_t) f_t^{1-\gamma} (x + g_t)^{\gamma-1} \\ & + (r + \nu_t) x f_t^{1-\gamma} (x + g_t)^{\gamma-1} - \frac{1}{1-\gamma} \frac{(\alpha-r)^2}{2\sigma^2} f_t^{1-\gamma} (x + g_t)^\gamma = 0, \end{aligned}$$

and obtain

$$g_t = \int_t^{\tilde{T}} s_{-t} p_{y+t} e^{-r(t-s)} b_s^{state} ds, \quad (21)$$

$$f_t = \int_t^{\tilde{T}} (s_{-t} \tilde{p}_{y+t})^{1/(1-\gamma)} (s_{-t} p_{y+t})^{-\gamma/(1-\gamma)} e^{\frac{\gamma}{1-\gamma} \varphi(t-s)} \left[w_s \left(1 + \left(k \frac{\mu_s}{\nu_s} \right)^{1/(1-\gamma)} \right) \right] ds, \quad (22)$$

where ${}_{s-t}p_{y+t} = e^{-\int_t^s \nu_\tau d\tau}$, ${}_{s-t}\tilde{p}_{y+t} = e^{-\int_t^s \mu_\tau d\tau}$, and $\varphi = r + \frac{(\alpha-r)}{2\sigma^2(1-\gamma)}$. The total optimal benefits and the optimal size of death sum are given by:

$$\begin{aligned} \frac{\partial}{\partial B} &: w^{1-\gamma}(B_t + b_t^{state})^{\gamma-1} - \frac{\partial V(t,x)}{\partial x} = 0 \\ \Rightarrow & B_t^* + b_t^{state} = \frac{w_t}{f_t}(x + g_t) = \frac{1}{\bar{a}_t^*}(x + g_t), \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{\partial}{\partial Beq} &: \mu_t k w_t^{1-\gamma} Beq_t^{\gamma-1} - \nu_t \frac{\partial V(t,x)}{\partial x} = 0 \\ \Rightarrow & Beq_t^* = \left(k \frac{\mu_t}{\nu_t}\right)^{1/(1-\gamma)} \frac{w_t}{f_t}(x + g_t) = \left(k \frac{\mu_t}{\nu_t}\right)^{1/(1-\gamma)} \frac{1}{\bar{a}_t^*}(x + g_t), \end{aligned} \quad (24)$$

where

$$\bar{a}_t^* = \int_t^{\tilde{T}} e^{-\int_t^s (\bar{\mu}_\tau + \bar{r}) d\tau} \left(1 + \left(k \frac{\mu_s}{\nu_s}\right)^{1/(1-\gamma)}\right) ds, \quad (25)$$

and $\bar{r} = \frac{1}{1-\gamma}\rho - \frac{\gamma}{1-\gamma}\varphi$ and $\bar{\mu}_t = \frac{1}{1-\gamma}\mu_t - \frac{\gamma}{1-\gamma}\nu_t$. The optimal proportion of the savings invested in the mutual fund is given by

$$\frac{\partial}{\partial \Pi} : (\alpha - r)x \frac{\partial V(t,x)}{\partial x} + \Pi_t \sigma^2 x^2 \frac{\partial^2 V(t,x)}{\partial x^2} = 0 \quad \Rightarrow \quad \Pi_t^* = \frac{\alpha-r}{(1-\gamma)\sigma^2} \frac{x+g_t}{x}, \quad (26)$$

and the proportions between the risky assets in the mutual fund are specified by the mutual fund theorem, see (Merton, 1969) and (Richard, 1975):

$$\forall_{i=1,\dots,N-1} \quad \theta_i = \frac{\sum_{j=1}^{N-1} [\sigma_{ij}]^{-1} (\alpha_j - r)}{\sum_{i=1}^{N-1} \sum_{j=1}^{N-1} [\sigma_{ij}]^{-1} (\alpha_j - r)}, \quad \sum_i \theta_i = 1. \quad (27)$$

Asset class	Real values		Correlations			
	Long-term rate	Volatility	Cash	Bonds	Dom. stocks	Int. stocks
Cash	0.7%	3.80%	1.00	0.30	-0.05	-0.03
Bonds	2.4%	7.10%		1.00	0.15	0.20
Dom. stocks	8.2%	19.7%			1.00	0.66
Int. stocks	7.3%	19.6%				1.00

	a_1	b_1	c_1	a_2	b_2	c_2
ν_t	-2.531	123.5	10.57	1.041e+15	660.7	93.88

Table 4: Statistical properties of the considered asset classes estimated as the historical real values, and constants for the mortality rate model.